[RW] Chapter 2.4: Induced Representations

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1 Bimodule ${}_A\mathsf{X}_B$

Definition 1.1. Let B be C*-algebras and let X_B be a right Hilbert B-bimodule. That is, X_B is a complex vector space with a right B-action satisfying:

$a \cdot x = \varphi(a)x$

(i) $(x + y) \cdot a = x \cdot a + y \cdot a$,

(ii) $x \cdot (a + b) = x \cdot a + x \cdot b$,

(iii) $(x \cdot a) \cdot b = x \cdot (ab)$,

(iv) $(\lambda x) \cdot a = x \cdot (\lambda a) = \lambda (x \cdot a)$,

for all $a, b \in B$, $x, y \in \mathsf{X}_B$, and $\lambda \in \mathbb{C}$.

In addition, X_B has an B-valued the inner product $\langle \cdot, \cdot \rangle_B : \mathsf{X}_B \times \mathsf{X}_B \to B$ satisfying

(i) $\langle x, \lambda y + \mu z \rangle_B = \lambda \langle x, y \rangle_B + \mu \langle x, z \rangle_B$,

(ii) $\langle x, y \cdot a \rangle_B = \langle x, y \rangle_B a$, $\longrightarrow$ ${}_A\langle a \cdot x, y \rangle = a\, {}_A\langle x, y \rangle$

(iii) $\langle x, y \rangle_B^* = \langle y, x \rangle_B$,

(iv) $\langle x, x \rangle_B \geq 0$ (as a positive element of B),

(v) $\langle x, x \rangle_B = 0$ implies that $x = 0$,

for all $x, y \in \mathsf{X}_B$, and $\lambda, \mu \in \mathbb{C}$.

Moreover, X_B is complete with respect to the norm defined by $\|x\|_B := \|\langle x, x \rangle_B\|^{1/2}$.

Definition 1.2. Let A be a C*-algebra, we can similarly define a left Hilbert A-module X_A using left-hand versions of the properties in Definition 1.1.

Let X_B be a Hilbert B-module and let A act as adjointable operators on X_B. That is, we have a C*-homomorphism of $\varphi : A \to \mathcal{L}(\mathsf{X}_B)$. We denote this left action of $a \in A$ on $x \in \mathsf{X}_B$ by $a \cdot x$. More explicitly, $a \cdot x = \varphi(a)(x)$. We denote the A-B-bimodule by ${}_A\mathsf{X}_B$.

representation of B $\pi : B \longrightarrow \mathcal{B}(\mathcal{H}_\pi)$

Goal: given a right Hilbert B-module X_B and an action of A on X_B as adjointable operators, we wish to convert a representation π of B on a Hilbert space $\mathcal{H}_\pi$ to a representation of A on some Hilbert space using the bimodule ${}_A\mathsf{X}_B$.

A

Motivating Example: Let G be a unimodular locally compact group with closed (unimodular) subgroup H. We will study a left action of $C^*(G)$ on the right Hilbert module $\mathsf{X}_{C^*(H)}$, B *inducing* representations of $C^*(H)$ "up" to representations of $C^*(G)$.

2 Induced representation

Notation: Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. We use $\mathcal{H} \otimes_{\mathrm{alg}} \mathcal{K}$ to denote the algebraic (i.e. incomplete) tensor product of $\mathcal{H}$ and $\mathcal{K}$ as vector spaces, and use $\mathcal{H} \otimes \mathcal{K}$ to denote the tensor product of Hilbert spaces. [RW] uses the notation $\mathcal{H} \odot \mathcal{K}$ for $\mathcal{H} \otimes_{\mathrm{alg}} \mathcal{K}$.

Setup: Let X_B be a Hilbert B-module A acting from the left as adjointable operators and let $\pi : B \to \mathcal{B}(\mathcal{H}_\pi)$ be a nondegenerate representation. We construct a representation of A on the interior tensor product of X_B and $\mathcal{H}_\pi$ (using π).

Definition 2.1. (interior tensor product of Hilbert C*-modules) Let E be a Hilbert A-module and F be a Hilbert B-module and let $\phi : A \to \mathcal{L}(F)$ be a C*-homomorphism. The interior tensor product $E \otimes_\phi F$ (using ϕ) is the completion of the underlying vector space

mod $\quad x a \otimes y = x \otimes \phi(a) y$

$$(E \otimes_{\mathrm{alg}} F) \cancel{/} \operatorname{span}\{xa \otimes_{\mathrm{alg}} y - x \otimes_{\mathrm{alg}} \phi(a)y : x \in E, y \in F, a \in A\}$$

with respect to the norm induced by the B-valued inner product

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle_B = \langle y_1, \phi(\langle x_1, x_2 \rangle_A) y_2 \rangle_B.$$

The interior tensor product $E \otimes_\phi F$ is a Hilbert B-module.

Example 2.2. By taking $E = \mathsf{X}_B$, $F = \mathcal{H}_\pi$, and $\phi = \pi$ in Definition 2.1, the interior tensor product of $\mathsf{X}_B \otimes_\pi \mathcal{H}_\pi$ (using π) is a right Hilbert $\mathbb{C}$-module (i.e., a Hilbert space). In [RW], $\mathsf{X}_B \otimes_\pi \mathcal{H}_\pi$ is denoted by $\mathsf{X} \otimes_B \mathcal{H}_\pi$, without explicitly mentioning interior tensor products. We adapt this notation.

Hilbert space

$\mathcal{H}$

Hilbert space

Explicitly, as a Hilbert space, the inner product on $\mathsf{X} \otimes_B \mathcal{H}_\pi$ is characterized by

$\langle ax+by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$

$$\langle x \otimes h, y \otimes k \rangle = \langle \pi(\langle y, x \rangle_B) h, k \rangle.$$

Hilbert $\mathbb{C}$-module

The tensor product is B-balanced in the sense that $(x \cdot b) \otimes h = x \otimes \pi(b)h$ for all $b \in B$. For emphasis of such manipulations, we write $x \otimes_B y \in \mathsf{X} \otimes_B \mathcal{H}_\pi$ for the image of $x \otimes y \in \mathsf{X}_B \otimes_{\mathrm{alg}} \mathcal{H}_\pi$.

$\langle a, b\rangle_{\mathbb{C}} = \langle b, a\rangle$

$(b|a)$ [RW]'s notation for Hilbert space inner product

$\pi : B \longrightarrow \mathcal{B}(\mathcal{H}_\pi)$

Proposition 2.3 (RW, Proposition 2.66)**.** *With the setup as above, the formula*

$\operatorname{Ind}\pi : A \longrightarrow \mathcal{B}(X \otimes_B \mathcal{H}_\pi)$

$$\boxed{\operatorname{Ind}\pi}(a)\,(x \otimes_B h) := (a \cdot x) \otimes_B h,$$

$\{a \cdot x : a \in A, x \in X\}$

when extended linearly, gives a representation $\operatorname{Ind}\pi$ *of* A *on the Hilbert space* $\mathsf{X} \otimes_B \mathcal{H}_\pi$*. If* X *is nondegenerate as an* A*-module (i.e.,* $A \cdot \mathsf{X}$ *is dense in* X*), then* $\operatorname{Ind}\pi$ *is a nondegenerate representation of* A *on* $\mathsf{X} \otimes_B \mathcal{H}_\pi$*.*

Notation. Since $\operatorname{Ind}\pi$ depends on both the Hilbert B-module X_B and the C*-homomorphism $A \to \mathcal{L}(\mathsf{X}_B)$ we are using, we write $\mathsf{X}\text{–}\operatorname{Ind}_B^A \pi$, $\operatorname{Ind}_B^A \pi$, or $\mathsf{X}\text{–}\operatorname{Ind}\pi$ for emphasis when needed.

$\varphi : A \longrightarrow \mathcal{L}(X_{\mathbb{C}}) = \mathcal{B}(\mathcal{H})$

Example 2.4. Take $B = \mathbb{C}$ and consider Hilbert space $\mathsf{X}_{\mathbb{C}} = \mathcal{H}$ and let $\rho : A \to \mathcal{B}(\mathcal{H})$ be a representation. Let $\pi : \mathbb{C} \to \mathcal{B}(\mathcal{H}_\pi)$ be given by $\pi(\lambda) = \lambda 1$, where 1 is the identity operator on $\mathcal{H}_\pi$. Then $\mathcal{H} \otimes_{\mathbb{C}} \mathcal{H}_\pi$ is the Hilbert space tensor product $\mathcal{H} \otimes \mathcal{H}_\pi$ and $\operatorname{Ind}\pi$ is given by $\operatorname{Ind}\pi = \rho \otimes 1$, that is, $\operatorname{Ind}\pi(a) = \rho(a) \otimes 1$.

Hilbert space tensor product

$\mathcal{H} \otimes X_{\mathbb{C}}$

$$\begin{aligned}
\langle x \otimes h, y \otimes k \rangle &= \langle \pi(\langle y, x\rangle_{\mathbb{C}})\, h, k \rangle \\
&= \langle \langle y, x\rangle_{\mathbb{C}}\, h, k \rangle \\
&= \langle x, y\rangle \langle h, k \rangle
\end{aligned}$$

$$\mathrm{id}\ \phi : \underset{\mathcal{L}(B_B)}{\overset{\shortparallel}{A}} \longrightarrow \mathcal{L}(B_B)$$

Example 2.5. Consider $\mathsf{X}_B = B_B$ and $A = \mathcal{L}(B_B) = M(B)$ and a nondegenerate representation $\pi : B \to \mathcal{H}_\pi$. There is a unitary map from $\mathcal{B} \otimes_B \mathcal{H}_\pi$ to $\mathcal{H}_\pi$, and the induced representation of $A = M(B)$ on $\mathcal{H}_\pi$ is the usual extended representation of a the multiplier algebra.

$\varphi : B \to \mathcal{B}(\mathcal{H}_\pi)$

extend $\tilde{\varphi} : M(B) \to \mathcal{B}(\mathcal{H}_\pi)$.

Induced representation is functorial: $\mathsf{X} - \operatorname{Ind}_B^A$ is a functor from the category of nondegenerate representations of B and bounded intertwining operators to the corresponding category for A. More precisely:

Proposition 2.6 (RW, Proposition 2.69)**.** *Suppose A acts nondegenerately as adjointable operators on a Hilbert B-module X, that*

$$\pi_1 : B \to \mathcal{B}(\mathcal{H}_1) \quad \textit{and} \quad \pi_2 : B \to \mathcal{B}(\mathcal{H}_2)$$

are nondegenerate representations of B. Let $T : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded intertwining operator:

$$T\left(\pi_1(b)h\right) = \pi_2(b)\left(Th\right).$$

Then the transformation

$$1 \otimes T : \begin{cases} \mathsf{X} \otimes_{\mathrm{alg}} \mathcal{H}_1 & \mapsto \quad \mathsf{X} \otimes_{\mathrm{alg}} \mathcal{H}_2 \\ x \otimes h & \mapsto \quad x \otimes (Th) \end{cases}$$

extends to a bounded operator $1 \otimes_B : \mathsf{X} \otimes_B \mathcal{H}_1 \mapsto \mathsf{X} \otimes_B \mathcal{H}_2$ which intertwines $\mathsf{X} - \operatorname{Ind} \pi_1$ and $\mathsf{X} - \operatorname{Ind} \pi 2$.

The correspondence $T \mapsto 1 \otimes_B T$ is $*$-linear, and if $S : \mathcal{H}_2 \to \mathcal{H}_3$ intertwines π_2 and π_3, then

$$1 \otimes (S \circ T) = (1 \otimes_B S) \circ (1 \otimes_B T).$$

3 Main example: ${}_{C^*(G)}\mathsf{X}_{C^*(H)}$

$C^*(G) \sim_{SME} C^*(H)$

3.1 (full) Group C*-algebra

$K[G]$

Let G be a unimodular locally compact group with Haar measure. $C_c(G)$ (the set of continuous $\mathbb{C}$-valued function on G with compact support) is a $*$-algebra under

G discrete

convolution
$$f * g = \int_G f(r) g\left(r^{-1}s\right) dr,$$

$C_c(G) = \mathbb{C}[G]$

$\sum'_{g \in G} a_g g$ $a_g = 0$ except for finite many $g \in G$.

and

involution
$$f^*(s) = \overline{f\left(s^{-1}\right)}.$$

$$\Big(\sum_{g\in G} a_g g\Big) * \Big(\sum_{g'\in G} b_{g'} g'\Big) = \sum_{h \in G}\Big(\sum_{g\in G} a_g b_{g^{-1}h}\Big) h$$

$$\Big(\sum_{g\in G} a_g g\Big)^* = \sum'_{g\in G} \overline{a_g}\, g^{-1}$$

Let $U : G \to \mathcal{U}(\mathcal{H})$ be a unitary presentation of G denoted by $s \mapsto U_s$ such that $s \mapsto U_s h$ from G to $\mathcal{H}$ is continuous for every $s \in G$ and $h \in \mathcal{H}$. For each unitary representation U,

$U : G \to U(\mathcal{H})$

$$\pi_U : \begin{cases} C_c(G) & \to \quad \mathcal{B}(\mathcal{H}) \\ f & \mapsto \quad \int_G f(s) U_s ds \end{cases}$$

$\pi_U\Big(\sum'_{g\in G} a_g g\Big) = \sum'_{g\in G} a_g U(g)$

is a $*$-representation of $C_c(G)$. In particular,

$$\langle \pi_U(f)h, k\rangle_{\mathcal{H}} = \int_G f(r)\langle U_r h, k\rangle dr$$

$\pi_U : C_c(G) \to \mathcal{B}(\mathcal{H})$

for all $f \in C_c(G)$, and $h, k \in \mathcal{H}$.

$U : G \to \mathcal{U}(\mathcal{H})$

$\pi_U : C_c(G) \to \mathcal{B}(\mathcal{H})$ nondegenerate

The map $U \mapsto \pi_U$ is a bijection between the unitary representations of G and the nondegenerate representations of $C_c(G)$.

Taking the closure of $C_c(G)$ with the universal norm:

$$\|f\|_{C^*(G)} = \sup\{\|\pi_U(f)\| : U \text{ if a unitary representation of } G\},$$

$C^*(G) := \overline{C_c(G)}^{C^*(G)}$ is a C*-algebra called the (full) group C*-algebra.

reduced $L^2(G)$

3.2 $_{C^*(G)}\mathsf{X}_{C^*(H)}$

Let H be a (unimodular) closed subgroup of G. We construct a right Hilbert $C^*(H)$-module $\mathsf{X}_{C^*(H)}$ from $C_c(G)$. For any $b \in C_c(H)$ and $f, g \in C_c(G)$, we define

$\|$

$\mathsf{X}_{C^*(H)}$

module
$$f \cdot b(s) = \int_H f\left(st^{-1}\right) b(t)dt$$

and

inner product
$$\langle f, g\rangle_{C_c(H)}(s) = \int_G \overline{f(r)}g(rs)dr.$$

With the appropriate completions, $C_c(G)$ as a right $C_c(H)$-module gives a right Hilbert $C^*(H)$-module $\mathsf{X}_{C^*(H)}$.

Define action of $C^*(G)$ on $\mathsf{X}_{C^*(H)}$. We first define an action of G on $C_c(G)$ (as a right $C_c(H)$-module): for $s \in G$, define $u_s : C_c(G) \to C_c(G)$ by

unitary "rep" of G

$$u_s(f)(t) := f\left(s^{-1}t\right)$$

for $f \in C_c(G)$ and $t \in G$. In particular, u_s is isometric in the sense that

$$\langle u_s(f), u_s(g)\rangle_{C_c(H)} = \langle f, g\rangle_{C_c(H)}$$

and u_s is adjointable with $u_s^* = u_{s^{-1}}$.

extend it to a "representation" of $C_c(G)$

For $z \in C_c(G) \subset C^*(G)$ and $f \in C_c(G) \subset \mathsf{X}_{C^*(H)}$, we define

$$(z \cdot f)(s) := \int_G z(r)u_r(f)(s)dr = \int_G z(r)f\left(r^{-1}s\right)dr.$$

This extends to a left action of $C^*(G)$ on the (complete) Hilbert $C^*(H)$-module $\mathsf{X}_{C^*(H)}$.

Induce representations of $C^*(H)$ to representations of $C^*(G)$ using $_{C^*(G)}\mathsf{X}_{C^*(H)}$.

$C(H X G/H)$, $\quad {}_H X_G$

Reference

[RW] I. Raeburn and D. P. Williams, *Morita Equivalence and Continuous-Trace C^*-Algebras*, Mathematical Surveys and Monographs no. 60, American Mathematical Society, Providence RI, 1998.